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On the regularity of the magnetic field in a diffusive plasma

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Abstract

It is known that magnetic fields in ideal chaotic plasmas tend to become extremely irregular and to concentrate in a fractal set, and it is assumed that the presence of a positive resistivity will have a smoothing effect. Here we try to quantify this effect by proving new inequalities which, on the one hand, relate the local and global size of velocity and magnetic field with the gradient of this field, and on the other provide a bound of the area of generalized level surfaces.

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1. Introduction

Most real plasmas are more or less turbulent, and yet their magnetic fields often have a recognizable large-scale structure. This is worth emphasizing because in an ideal plasma (without resistivity) the magnetic field lines are transported by the fluid flow as material points, so we would expect that a chaotic flow will tend to produce an extremely tangled and convoluted geometry of field lines. This indeed has been shown to be the case both in theoretical models and in simulations [1, 2]. It has been found that the field tends to concentrate in a fractal set of zero measure whose dimension is determined by the Lyapunov coefficients of the flow and that the field direction follows approximately the most unstable direction of the flow. In this way a sheet-like or rope-like structure develops, according to the number of negative Lyapunov exponents [2, 3]. This is relevant because fast dynamos (those that increase exponentially for some time [4, 5]) require chaotic flows and a tentative formula for the growth rate has been proposed [3] (and criticized [6]). This formula involves the Lyapunov coefficients and a certain cancellation exponent which essentially measures how rapidly the field oscillates transversally to the main direction, which happens infinitely more often in the ideal case. These results are not totally rigorous, but they appear in many models and seem rather robust: concentration of the field in sets of progressively smaller measure is well known and related to intermittency [7]. This has been thoroughly analysed for some chaotic flows, such as the ABC one [8]. Although most of the studies are kinematic, i.e. the influence of the field upon the velocity through the Lorentz force is ignored, this is perfectly reasonable as long as the field is small; anyway this is the way to proceed for results which do not depend on any specific form of velocity. All plasmas, however, have a positive if very small resistivity, which allows for reconnection and changes in the field line geometry. The fact that this diffusivity factor is small and nevertheless not only reconnection of field lines occurs, but it is responsible for the release of enormous amounts of magnetic energy in such spectacular phenomena as solar flares [9], has made several authors to hypothesize that resistivity is enhanced at the zones of fast reconnection by different processes not comprised within the framework of standard magnetohydrodynamics. We will not make such assumptions and will not specify the geometry of these zones; instead we take as usual a constant resistivity throughout the domain and a generic velocity field. Of course it is known that the introduction of positive resistivity makes the induction equation governing the evolution of the magnetic field a parabolic one and therefore ensures to some extent smoothness of the solutions. But a vector field may be smooth and nevertheless have a very complex field line geometry. We intend to show that, no matter how chaotic the plasma flow is, the magnetic field is rather well behaved. The key to obtaining the necessary estimates is the analysis of the induction equation satisfied by any convex function of the field. This technique was introduced by Constantin [10, 11] to study the vorticity and other magnitudes of a fluid flow. The magnetic field, however, although formally satisfying an analogous equation, behaves in a different way: neither initial values nor boundary conditions are tied to the velocity and so its possible configurations are much wider. Also the emphasis is different: active regions of a plasma, where the magnetic field is large, are more important than regions of high vorticity in a fluid, and magnetic field lines and level surfaces are physically more meaningful and interesting than the analogous concepts for the vorticity.

The plan of the paper is as follows: in section 2 we study the basic induction equation satisfied by a convex function Φ of the magnetic field $\boldsymbol{B}$ and its time and space integrals. This equation yields two different sets of estimates, according to the term one emphasizes. The first set bounds the quadratic term involving the second derivatives of Φ and finds some relevant information on the mean size of the gradient of $\boldsymbol{B}$. In section 3 we localize the induction equation and make analogous estimates within a ball in the domain, extracting consequences on the ideal picture of a wildly oscillant magnetic field. The second possibility is to consider the quadratic term simply as positive and bound the remaining equation, a procedure followed in section 4. This approach is particularly useful for studying the area of generalized level surfaces: sets of the form $\Phi(\boldsymbol{B}) = \text{const}$, which include the classical level surfaces of constant magnitude as a particular case. These areas are bounded with surprising independence of Φ, indicating that the field geometry cannot be too complicated. Finally, the results are summarized in section 5.

2. Global estimates

The magnetic field within an incompressible plasma of velocity $\boldsymbol{u}$ and resistivity η satisfies, in the magnetohydrodynamic (MHD) approximation, the induction equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla - \eta\Delta\right)\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u} \tag{1}$$

plus some initial and boundary conditions. We will always assume that the fluid does not cross the (smooth) boundary of the domain under consideration, i.e. $\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$, and that the normal component of the magnetic field also vanishes there: $\boldsymbol{B}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$. These are the only hypotheses; obviously the final estimates will be better if the velocity has good boundedness properties, but this is not necessary for our argument.

Let Φ be a smooth nonnegative convex function defined in the whole space (or simply in any open set containing the range of $\boldsymbol{B}$) such that we know *a priori* that $\Phi \circ \boldsymbol{B}$ (i.e. the composition of both functions) decreases towards the boundary, in the sense that $\partial(\Phi \circ \boldsymbol{B})/\partial n|_{\partial\Omega} \leqslant 0$. This may be guaranteed, for instance, if $\boldsymbol{B}$ satisfies a Dirichlet condition $\boldsymbol{B}|_{\partial\Omega} = \mathbf{0}$ and $\Phi(\mathbf{0}) = 0$. Another possibility is as follows: since usually one studies the domain where the action takes place and outside which the magnetic field is smaller, it often occurs that $\partial B^2/\partial n|_{\partial\Omega} \leqslant 0$. Then any function $\Phi(\boldsymbol{B}) = g(B^2)$, with g a positive convex function, satisfies the requirement.

After some cumbersome, but essentially straightforward, integral inequalities (appendix A), one finds

$$\frac{1}{T}\int_0^T\int_\Omega \nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{2}{\eta T}\int_\Omega \Phi\circ\boldsymbol{B}(0)\,\mathrm{d}V + \frac{1}{\eta^2 T}\int_0^T\int_\Omega |\boldsymbol{B}|^2\boldsymbol{u}\cdot(\Phi''\circ\boldsymbol{B})\cdot\boldsymbol{u}\,\mathrm{d}V\,\mathrm{d}t. \tag{2}$$

Now, the first term on the right-hand side is divided by T and therefore grows increasingly irrelevant with time. The interest of this inequality is that it provides a collection of bounds on $\nabla\boldsymbol{B}$, depending on the convex funtion Φ. The simplest bound is obtained from $\Phi(\boldsymbol{x}) = x^2$, yielding $\Phi'' = 2I$; this shows that the L^2-norm of $\nabla\boldsymbol{B}$, measuring the quadratic mean of the gradient of the magnetic field, cannot be too large if the intensity of the field and velocity are not (for instance, if the L^4-norms of them are bounded). Note that this is not a classical energy inequality [14], which demands the use of the full MHD system including the momentum equation and obtains a different scaling.

Obviously the advantages of (2) are due to the flexibility on the choosing of Φ. As an example, assume that B^2 decreases towards the boundary. Then take $\Phi(\boldsymbol{x}) = g(x^2)$, $g' > 0$, $g'' > 0$. Then (2) becomes (appendix B)

$$\frac{1}{T}\int_0^T\int_\Omega 2g'(B^2)|\nabla\boldsymbol{B}|^2 + g''(B^2)|\nabla(B^2)|^2\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{2}{\eta T}\int_\Omega g(B(0)^2)\,\mathrm{d}V + \frac{1}{\eta^2 T}\int_0^T\int_\Omega B^2(2g'(B^2)|\boldsymbol{u}|^2 + 4g''(B^2)(\boldsymbol{u}\cdot\boldsymbol{B})^2)\,\mathrm{d}V. \tag{3}$$

The previous inequality yields different insights according to the larger differential of g. For instance, if one takes $g(x) = \exp(\alpha x)$, with $\alpha > 0$ large, $g'(B^2) = \alpha g(B^2)$, $g''(B^2) = \alpha^2 g(B^2)$. Hence the largest factor is $g''(B^2) = \alpha^2\exp(\alpha B^2)$. If we omit the remaining terms, the inequality becomes approximately

$$\frac{1}{T}\int_0^T\int_\Omega \exp(\alpha B^2)|\nabla(B^2)|^2\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{4}{\eta^2 T}\int_0^T\int_\Omega B^2\exp(\alpha B^2)(\boldsymbol{u}\cdot\boldsymbol{B})^2\,\mathrm{d}V\,\mathrm{d}t. \tag{4}$$

Assume that velocity and field are nearly orthogonal in the domain. Then the right-hand integral is small and so must the left-hand one be, so B should be almost constant in Ω. The assumption may seem excessive, but it may occur in certain cases: if we study the alfvénic oscillations of an equlibrium magnetic field, the velocity is orthogonal to the equilibrium field, while the product of the perturbed velocity and field is quadratically small compared to the size of the perturbations. Hence for the alfvénic oscillations to represent the largest part of the perturbed velocity, the equilibrium field must be roughly constant in size. As another example, several simplified models consider a plasma flow in a plane with an orthogonal magnetic field [12].

Another nontrivial application is that the variation of any component of the field may be bounded by the same component of the velocity. Take for simplicity the component B_1, and choose $\Phi(\boldsymbol{x}) = g(x_1)$, $g'' > 0$. We are left with

$$\frac{1}{T}\int_0^T\int_\Omega g''(B_1)|\nabla B_1|^2\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{2}{\eta T}\int_\Omega g(B_1(0))\,\mathrm{d}V + \frac{1}{\eta^2 T}\int_0^T\int_\Omega B^2 g''(B_1)u_1^2\,\mathrm{d}V\,\mathrm{d}t \tag{5}$$

so that the only influence of the remaining velocity components is through the term B^2, which of course depends on $\boldsymbol{u}$. Of course the same may be done with $\boldsymbol{B}\cdot\boldsymbol{e}$ for any constant vector $\boldsymbol{e}$.

3. Local estimates

The previous technique may be applied to study the local structure of the field. Take as Φ any nonnegative convex function without bothering for the behaviour of $\Phi\circ\boldsymbol{B}$ at the boundary, and let $f \geqslant 0$ be any time-independent, smooth function whose support is contained within Ω. By an argument similar to the previous ones (appendix C), one gets

$$\begin{aligned}\frac{\eta}{2}\int_0^T\int_\Omega & f\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}\,\mathrm{d}V\,\mathrm{d}t \\ &\leqslant \int_\Omega f\Phi(\boldsymbol{B}(0))\,\mathrm{d}V + \frac{1}{2\eta}\int_0^T\int_\Omega fB^2\boldsymbol{u}\cdot\Phi''(\boldsymbol{B})\cdot\boldsymbol{u}\,\mathrm{d}V\,\mathrm{d}t \\ &\quad + \left|\int_0^T\int_\Omega(\Phi\circ\boldsymbol{B})(\boldsymbol{u}\cdot\nabla f + \eta\Delta f)\,\mathrm{d}V\,\mathrm{d}t\right| \\ &\quad + \left|\int_0^T\int_\Omega(\boldsymbol{B}\cdot\nabla f)(\nabla\Phi(\boldsymbol{B})\cdot\boldsymbol{u})\,\mathrm{d}V\,\mathrm{d}t\right|.\end{aligned} \tag{6}$$

We now particularize f and Φ. Take a point $\boldsymbol{x}_0 \in \Omega$ such that the ball B_{2r} of centre $\boldsymbol{x}_0$ and radius $2r$ is contained in Ω. Let f be such that its value is 1 at B_r, its support is contained in B_{2r}, $|\nabla f| \leqslant 2/r$, $|\Delta f| \leqslant 6/r^2$. As for Φ, since we are working in a small ball, its global behaviour does not matter too much. We simply set $\Phi(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{B}_0|^2$, where $\boldsymbol{B}_0 = \boldsymbol{B}(\boldsymbol{x}_0)$; of course $\boldsymbol{B}_0$ depends on time. Then $\nabla\Phi(\boldsymbol{B}) = 2(\boldsymbol{B} - \boldsymbol{B}_0)$, $\Phi'' = 2I$. Therefore

$$\begin{aligned}\frac{1}{T}\int_0^T\int_{B_r}|\nabla\boldsymbol{B}|^2\,\mathrm{d}V\,\mathrm{d}t &\leqslant \frac{2}{\eta T}\int_{B_{2r}}|\boldsymbol{B}(0) - \boldsymbol{B}_0(0)|^2\,\mathrm{d}V + \frac{1}{\eta^2 T}\int_0^T\int_{B_{2r}}B^2u^2\,\mathrm{d}V\,\mathrm{d}t \\ &\quad + \frac{4}{\eta r T}\int_0^T\int_{B_{2r}}u|\boldsymbol{B} - \boldsymbol{B}_0|^2\,\mathrm{d}V\,\mathrm{d}t + \frac{12}{r^2 T}\int_0^T\int_{B_{2r}}|\boldsymbol{B} - \boldsymbol{B}_0|^2\,\mathrm{d}V\,\mathrm{d}t \\ &\quad + \frac{8}{\eta r T}\int_0^T\int_{B_{2r}}uB|\boldsymbol{B} - \boldsymbol{B}_0|\,\mathrm{d}V\,\mathrm{d}t.\end{aligned} \tag{7}$$

To understand the implications of this formula, we must realize that typically $|\boldsymbol{B} - \boldsymbol{B}_0|$ scales like Mr. Unless larger and larger jumps of $\boldsymbol{B}$ exist within the ball, M is bounded through time. Note that in the right-hand side integrals there are no gradients and we may assume that any kind of mean of u and B within the ball has an order of Nr^3, with N bounded in time: i.e. there are no unbounded concentrations of velocity or magnetic flux within the ball. Therefore, we now assume that $\boldsymbol{B}$ and $\boldsymbol{u}$ remain bounded in the ball for all time. Since the small parameters are normally η and r, we are left with a bound

$$\frac{1}{T}\int_0^T\int_{B_r}|\nabla\boldsymbol{B}|^2\,\mathrm{d}V\,\mathrm{d}t \leqslant C\frac{r^3}{\eta^2} \tag{8}$$

so that the integral of $|\nabla\boldsymbol{B}|^2$ scales at most as r^3/η^2. Since the volume of B_r scales like r^3, we may rearrange (8) by

$$\left(\frac{1}{T}\int_0^T\frac{1}{\mathrm{Vol}(B_r)}\int_{B_r}|\nabla\boldsymbol{B}|^2\,\mathrm{d}V\,\mathrm{d}t\right)^{1/2} \leqslant \frac{C}{\eta}. \tag{9}$$

Let us compare this formula with the classical energy inequality. When the normal components of $\boldsymbol{u}$ and $\boldsymbol{B}$ vanish at the boundary, and both remain bounded (say by M) in Ω for $t \in [0, T]$, one gets by elementary integrations

$$\frac{1}{T}\int_0^T\int_\Omega |\nabla \boldsymbol{B}|^2 \,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{1}{T\eta}\int_\Omega B(0)^2\,\mathrm{d}V + \frac{M^4\mathrm{Vol}(\Omega)}{\eta^2}. \tag{10}$$

The first term of the right-hand side becomes irrelevant for large T. The second one is analogous to (9), but it represents an integral over the whole domain. Thus we have proved that the classical bound showing that the gradient of $\boldsymbol{B}$ within Ω has at most an order $1/\eta$, may be refined to account for local means of $\boldsymbol{u}$ and $\boldsymbol{B}$. Hence it cannot occur that in a certain portion of the domain the field may oscillate wildly, while making up for this behaviour by being almost constant in the rest of the domain.

4. Generalized level surfaces

Now we will not concentrate on the quadratic term in $\nabla \boldsymbol{B}$, but on the remaining portion of the equation. Let $F = \Phi \circ \boldsymbol{B}$. Our purpose is to analyse the mean area of the level surfaces $F = \Phi(\boldsymbol{B}) = \text{const}$. Let S_r be the set $F = \Phi(\boldsymbol{B}) = r$, $r \geqslant 0$. It is known [13] that for almost every r, S_r is a smooth surface (varying with time). For a certain r, S_r can fail to be a surface and may fill an open set of Ω, but this may only happen for a set of r's of measure zero. It is also known that if G is continuous in Ω,

$$\int_\Omega G|\nabla F|\,\mathrm{d}V = \int_0^\infty \mathrm{d}r \int_{S_r} G\,\mathrm{d}\sigma \tag{11}$$

where $\mathrm{d}\sigma$ represents the measure of area. Let ϕ be any arbitrary smooth function defined in $[0, \infty)$ and vanishing outside a bounded interval. Assume that the L^2-norm of $\nabla \boldsymbol{u}$ remains bounded in time. Let M_Φ denote the maximum of $|\nabla\Phi|$ within the range of $\boldsymbol{B}$. Then one may prove (appendix D) that

$$\frac{1}{T}\int_0^T\int_\Omega \phi(F)^2|\nabla F|^2\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{C}{\eta} M_\Phi \|\phi\|_2^2 \tag{12}$$

where C is a constant. Take $G = \phi \circ F$, which is constant in S_r. Then

$$\int_\Omega \phi(F)|\nabla F|\,\mathrm{d}V = \int_0^\infty \phi(r)A(S_r)\,\mathrm{d}r \tag{13}$$

and, denoting by $A_T(S_r)$ the mean area

$$A_T(S_r) = \frac{1}{T}\int_0^T A(S_r)\,\mathrm{d}t$$

we get

$$\frac{1}{T}\int_0^T\int_\Omega \phi(F)|\nabla F|\,\mathrm{d}V = \int_0^\infty \phi(r)A_T(S_r)\,\mathrm{d}r. \tag{14}$$

By the inequality of Cauchy–Schwarz,

$$\frac{1}{T}\int_0^T\int_\Omega \phi(F)|\nabla F|\,\mathrm{d}V\,\mathrm{d}t \leqslant \mathrm{Vol}(\Omega)^{1/2}\left(\frac{1}{T}\int_0^T\int_\Omega \phi(F)^2|\nabla F|^2\,\mathrm{d}V\,\mathrm{d}t\right)^{1/2}$$
$$\leqslant \left(\frac{\mathrm{Vol}(\Omega)CM_\Phi}{\eta}\right)^{1/2}\|\phi\|_2 \tag{15}$$

where $\mathrm{Vol}(\Omega)$ denotes as before the volume of Ω. The inequality

$$\int_0^\infty \phi(r) A_T(S_r)\,\mathrm{d}r \leqslant \left(\frac{\mathrm{Vol}(\Omega) C M_\Phi}{\eta}\right)^{1/2} \|\phi\|_2 \tag{16}$$

valid for a set of test functions ϕ dense in $L^2[0, \infty)$, yields

$$\int_0^\infty A_T(S_r)^2\,\mathrm{d}r \leqslant \frac{\mathrm{Vol}(\Omega) C M_\Phi}{\eta}. \tag{17}$$

Although this is a bound on the mean area of S_r, the integral is also influenced by how rapidly Φ varies, so that for the same geometrical surfaces one function could have a much longer interval of variation than another one and the integral on the left side would be correspondingly larger. Take for instance the usual level sets $B^2 = \text{const}$. They are the generalized level surfaces associated to $\Phi(\boldsymbol{x}) = x^2$, but also to $\Phi(\boldsymbol{x}) = h(x^2)$, for any increasing and convex h. If h increases rapidly the integral would be larger without a change in the surfaces themselves. A possible way to compensate this is to divide by M_Φ, which normalizes the integral in a sense, at least for functions whose gradient does not vary too much in the range of possible $\boldsymbol{B}$'s. Thus

$$\frac{1}{M_\Phi}\int_0^\infty A_T(S_r)^2\,\mathrm{d}r \leqslant \frac{m(\Omega) C}{\eta} \tag{18}$$

which is our main bound in this section. It says that the level surfaces cannot be too complicated in the mean and therefore to have a large area. Note that the right-hand term does not depend on Φ.

5. Conclusions

The study of the evolution equation satisfied by any convex function of the magnetic field within a diffusive plasma provides a new estimate on the integral of the gradient of the field, which includes as a particular case the classical energy inequality. By particularizing this convex function to those depending on the field magnitude, new relations with the behaviour of the field gradient and the orthogonality of field and velocity emerge. These estimates may also be obtained locally, and prove that the variation of the field within a ball may be bounded by quantities depending only on the local size of the magnetic field and velocity. The same procedure allows us to bound the mean area of generalized level surfaces. All these results tend to show that the field is rather regular for a positive diffusivity, even if the plasma flow is chaotic, so that large-scale structures may be recognizable.

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Appendix A

Let $F = \Phi \circ \boldsymbol{B}$. By elementary operations,

$$\begin{aligned}
\frac{\partial F}{\partial t} &= (\nabla\Phi \circ \boldsymbol{B}) \frac{\partial \boldsymbol{B}}{\partial t} \\
\boldsymbol{u} \cdot \nabla F &= (\nabla\Phi \circ \boldsymbol{B}) \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{B}) \\
\Delta F &= (\nabla\Phi \circ \boldsymbol{B}) \cdot \Delta \boldsymbol{B} + \nabla \boldsymbol{B} \cdot (\Phi'' \circ \boldsymbol{B}) \cdot \nabla \boldsymbol{B}
\end{aligned} \tag{19}$$

where the last term means $\sum_{i,j,k}(\partial^2_{j,k}\Phi \circ \boldsymbol{B})\partial_i B_j \partial_i B_k$.

Therefore F satisfies the equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla - \eta\Delta\right)F = (\nabla\Phi\circ\boldsymbol{B})\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla - \eta\Delta\right)\boldsymbol{B} - \eta\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}$$
$$= (\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u}) - \eta\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}. \tag{20}$$

Let us integrate in Ω all the terms of the equation. Since $\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$,

$$\int_\Omega \boldsymbol{u}\cdot\nabla F\,\mathrm{d}V = \int_{\partial\Omega} F\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0. \tag{21}$$

Also

$$\int_\Omega \frac{\partial F}{\partial t}\,\mathrm{d}V = \frac{\partial}{\partial t}\int_\Omega F\,\mathrm{d}V \tag{22}$$

$$\int_\Omega \eta\nabla F\,\mathrm{d}V = \eta\int_\Omega \frac{\partial F}{\partial n}\,\mathrm{d}\sigma \leqslant 0. \tag{23}$$

As for the integral of $(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u})$, it may be bounded in several ways. For the present purpose we use $\boldsymbol{B}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$ to get

$$\int_\Omega (\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u})\,\mathrm{d}V = -\int_\Omega \boldsymbol{B}\cdot\nabla(\nabla\Phi\circ\boldsymbol{B})\cdot\boldsymbol{u}\,\mathrm{d}V. \tag{24}$$

The right-hand integrand is

$$\sum_{i,j} B_j\partial_j(\partial_i\Phi\circ\boldsymbol{B})u_i = \sum_{i,j,k} B_j\left(\partial^2_{k,i}\Phi\circ\boldsymbol{B}\right)(\partial_j B_k)u_i = \sum_j B_j\boldsymbol{u}\cdot(\Phi''\circ\boldsymbol{B})\cdot\partial_j\boldsymbol{B}. \tag{25}$$

Let us denote by $b(\boldsymbol{v},\boldsymbol{w})$ the positive bilinear form given at a certain point $\boldsymbol{x}$ by the matrix $\Phi''(\boldsymbol{B}(\boldsymbol{x}))$, and let $\|\ \|_\Phi$ be its associated seminorm. By the Cauchy–Schwarz's inequality,

$$\left|\sum_j B_j b(\boldsymbol{u},\partial_j\boldsymbol{B})\right| \leqslant \sum_j |B_j|\|\boldsymbol{u}\|_\Phi\|\partial_j\boldsymbol{B}\|_\Phi \leqslant \sum_j \frac{\eta}{2}\|\partial_j\boldsymbol{B}\|^2_\Phi + \frac{1}{2\eta}|B_j|^2\|\boldsymbol{u}\|^2_\Phi \tag{26}$$

so that the integral of the right-hand side of (24) is bounded by

$$-\frac{\eta}{2}\int_\Omega\sum_j\|\partial_j\boldsymbol{B}\|^2_\Phi\,\mathrm{d}V + \frac{1}{2\eta}\int_\Omega|\boldsymbol{B}|^2\|\boldsymbol{u}\|^2_\Phi\,\mathrm{d}V. \tag{27}$$

Integrating in also time equation (20) and using the fact that $\Phi \geqslant 0$,

$$\frac{1}{T}\int_0^T\int_\Omega \nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}\,\mathrm{d}V\,\mathrm{d}t$$
$$\leqslant \frac{2}{\eta T}\int_\Omega \Phi\circ\boldsymbol{B}(0)\,\mathrm{d}V + \frac{1}{\eta^2 T}\int_0^T\int_\Omega |\boldsymbol{B}|^2\boldsymbol{u}\cdot(\Phi''\circ\boldsymbol{B})\cdot\boldsymbol{u}\,\mathrm{d}V\,\mathrm{d}t \tag{28}$$

which is the desired inequality.

Appendix B

Since

$$\partial^2_{i,j}\Phi(\boldsymbol{x}) = 4g''(x^2)x_i x_j + 2g'(x^2)\delta_{i,j} \tag{29}$$

we have

$$\Phi''(\boldsymbol{B}) = 2g'(B^2)I + 4g''(B^2)B_iB_j$$

$$\nabla\boldsymbol{B}\cdot\Phi''(\boldsymbol{B})\cdot\nabla\boldsymbol{B} = 2g'(B^2)|\nabla\boldsymbol{B}|^2 + 4g''(B^2)\sum_{i,j,k}B_iB_j\partial_kB_i\partial_kB_j$$
$$= 2g'(B^2)|\nabla\boldsymbol{B}|^2 + g''(B^2)|\nabla(B^2)|^2. \tag{30}$$

Also

$$\boldsymbol{u}\cdot\Phi''(\boldsymbol{B})\cdot\boldsymbol{u} = 2g'(B^2)|\boldsymbol{u}|^2 + 4g''(B^2)\sum_{i,j}B_iB_ju_iu_j = 2g'(B^2)|\boldsymbol{u}|^2 + 4g''(B^2)(\boldsymbol{u}\cdot\boldsymbol{B})^2. \tag{31}$$

Thus

$$\frac{1}{T}\int_0^T\int_\Omega 2g'(B^2)|\nabla\boldsymbol{B}|^2 + g''(B^2)|\nabla(B^2)|^2\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{2}{\eta T}\int_\Omega g(B(0)^2)\,\mathrm{d}V$$
$$+\frac{1}{\eta^2T}\int_0^T\int_\Omega B^2(2g'(B^2)|\boldsymbol{u}|^2 + 4g''(B^2)(\boldsymbol{u}\cdot\boldsymbol{B})^2)\,\mathrm{d}V. \tag{32}$$

Appendix C

Define $F = f(\Phi\circ\boldsymbol{B})$. We have now

$$\begin{aligned}\frac{\partial F}{\partial t} &= f(\nabla\Phi\circ\boldsymbol{B})\frac{\partial\boldsymbol{B}}{\partial t}\\ \boldsymbol{u}\cdot\nabla F &= (\Phi\circ\boldsymbol{B})\boldsymbol{u}\cdot\nabla f + f(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{B})\\ \Delta F &= (\Phi\circ\boldsymbol{B})\Delta f + 2\nabla f\cdot((\nabla\Phi\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B})\\ &\quad + f(\nabla\Phi\circ\boldsymbol{B})\cdot\Delta\boldsymbol{B} + f\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}.\end{aligned} \tag{33}$$

And therefore

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla-\eta\Delta\right)F = f(\nabla\Phi\circ\boldsymbol{B})\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla-\eta\Delta\right)\boldsymbol{B} + (\Phi\circ\boldsymbol{B})(\boldsymbol{u}\cdot\nabla f)$$
$$-\eta(\Phi\circ\boldsymbol{B})\Delta f - 2\eta\nabla f\cdot((\nabla\Phi\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}) - \eta f\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}$$
$$= f(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u}) + (\Phi\circ\boldsymbol{B})(\boldsymbol{u}\cdot\nabla f)$$
$$-\eta(\Phi\circ\boldsymbol{B})\Delta f - \eta f\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}. \tag{34}$$

Since F has compact support contained in Ω, the integral of the left-hand term is

$$\frac{\partial}{\partial t}\int_\Omega F\,\mathrm{d}V.$$

Also,

$$\int_\Omega(\Phi\circ\boldsymbol{B})\Delta f\,\mathrm{d}V = -\int_\Omega\nabla f\cdot((\nabla\Phi\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B})\,\mathrm{d}V$$

so that, integrating the whole equation,

$$\eta\int_\Omega f\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}\,\mathrm{d}V = -\frac{\partial}{\partial t}\int_\Omega F\,\mathrm{d}V + \int_\Omega(\Phi\circ\boldsymbol{B})(\boldsymbol{u}\cdot\nabla f + \eta\Delta f)$$
$$+ f(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u})\,\mathrm{d}V. \tag{35}$$

The term

$$\int_\Omega f(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u})\,\mathrm{d}V$$

equals

$$-\int_\Omega \boldsymbol{B}\cdot\nabla(f\nabla\Phi\circ\boldsymbol{B})\cdot\boldsymbol{u}\,\mathrm{d}V = -\int_\Omega f\sum_j B_j\boldsymbol{u}\cdot\Phi''(\boldsymbol{B})\cdot\partial_j\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla f)(\nabla\Phi(\boldsymbol{B})\cdot\boldsymbol{u})\,\mathrm{d}V. \tag{36}$$

Hence

$$\eta\int_\Omega f\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}\,\mathrm{d}V + \int_\Omega f\sum_j B_j\boldsymbol{u}\cdot\Phi''(\boldsymbol{B})\cdot\partial_j\boldsymbol{B}\,\mathrm{d}V = -\frac{\partial}{\partial t}\int_\Omega F\,\mathrm{d}V$$
$$+\int_\Omega(\Phi\circ\boldsymbol{B})(\boldsymbol{u}\cdot\nabla f+\eta\Delta f)\,\mathrm{d}V - \int_\Omega(\boldsymbol{B}\cdot\nabla f)(\nabla\Phi(\boldsymbol{B})\cdot\boldsymbol{u})\,\mathrm{d}V. \tag{37}$$

Using as before the inequality of Cauchy–Schwarz, now with the matrix $f\Phi''(\boldsymbol{B})$, we get

$$\left|\int_\Omega f\sum_j B_j\boldsymbol{u}\cdot\Phi''(\boldsymbol{B})\cdot\partial_j\boldsymbol{B}\,\mathrm{d}V\right|$$
$$\leqslant \frac{\eta}{2}\int_\Omega f\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}\,\mathrm{d}V + \frac{1}{2\eta}\int_\Omega fB^2\boldsymbol{u}\cdot\Phi''(\boldsymbol{B})\cdot\boldsymbol{u}\,\mathrm{d}V. \tag{38}$$

Integrating in time and using the above inequality,

$$\frac{\eta}{2}\int_0^T\int_\Omega f\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B}\,\mathrm{d}V\,\mathrm{d}t$$
$$\leqslant \int_\Omega f\Phi(\boldsymbol{B}(0))\,\mathrm{d}V + \frac{1}{2\eta}\int_0^T\int_\Omega fB^2\boldsymbol{u}\cdot\Phi''(\boldsymbol{B})\cdot\boldsymbol{u}\,\mathrm{d}V\,\mathrm{d}t$$
$$+\left|\int_0^T\int_\Omega(\Phi\circ\boldsymbol{B})(\boldsymbol{u}\cdot\nabla f+\eta\Delta f)\,\mathrm{d}V\,\mathrm{d}t\right|$$
$$+\left|\int_0^T\int_\Omega(\boldsymbol{B}\cdot\nabla f)(\nabla\Phi(\boldsymbol{B})\cdot\boldsymbol{u})\,\mathrm{d}V\,\mathrm{d}t\right| \tag{39}$$

which is the stated formula.

Appendix D

Let ψ be any increasing smooth function defined in $[0,\infty)$ and take $H=\psi\circ F$. As before, we have

$$\frac{\partial H}{\partial t} = \psi'(F)\frac{\partial F}{\partial t}$$
$$\boldsymbol{u}\cdot\nabla H = \psi'(F)(\boldsymbol{u}\cdot\nabla F) \tag{40}$$
$$\Delta H = \psi'(F)\Delta F + \psi''(F)|\nabla F|^2.$$

Therefore H satisfies

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla-\eta\Delta\right)H = \psi'(F)\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla-\eta\Delta\right)F - \eta\psi''(F)|\nabla F|^2$$
$$=\psi'(F)(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u}) - \eta\psi'(F)\nabla\boldsymbol{B}\cdot(\Phi''\circ\boldsymbol{B})\cdot\nabla\boldsymbol{B} - \eta\psi''(F)|\nabla F|^2$$
$$\leqslant \psi'(F)(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u}) - \eta\psi''(F)|\nabla F|^2. \tag{41}$$

In the following argument we will use a bound on the L^2 norm of $\nabla\boldsymbol{u}$, which we had avoided so far, but that for our present purpose seems simpler. We could as well do as in the previous proofs and reduce the integral of $(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u})$ to a term involving $B^2\boldsymbol{u}\cdot\Phi''(\boldsymbol{B})\cdot\boldsymbol{u}$, but now we are not as interested as before in avoiding all gradients for our estimate. Note that in three dimensions the only magnitudes which are *a priori* bounded are the kinetic and magnetic energy, i.e. the L^2 norms of $\boldsymbol{u}$ and $\boldsymbol{B}$ [14], although in dimension 2 the same is true of the enstrophy (the L^2-norm of $\nabla\boldsymbol{u}$). Recall that the size of $\nabla\boldsymbol{u}$ has little to do with the chaotic character of the flow.

Integrating the above inequality in Ω and assuming as before that $\partial F/\partial n \leqslant 0$, which implies $\partial H/\partial n \leqslant 0$, we are left with

$$\frac{\partial}{\partial t}\int_\Omega H\,\mathrm{d}V + \eta\int_\Omega \psi''(F)|\nabla F|^2\,\mathrm{d}V \leqslant \int_\Omega \psi'(F)(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u})\,\mathrm{d}V \tag{42}$$

i.e.

$$\frac{1}{T}\int_0^T\int_\Omega \psi''(F)|\nabla F|^2\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{1}{\eta T}\int_\Omega \psi(F(0))\,\mathrm{d}V + \frac{1}{\eta T}\int_0^T\int_\Omega \psi'(F)(\nabla\Phi\circ\boldsymbol{B})\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u})\,\mathrm{d}V\,\mathrm{d}t. \tag{43}$$

The integrand of the last integral is bounded by $\psi'(F)M_\Phi B|\nabla\boldsymbol{u}|$.

Let us take as ψ the second primitive of the square of the arbitrary compact-supported function ϕ,

$$\psi(s) = \int_0^s (s-v)\phi(v)^2\,\mathrm{d}v. \tag{44}$$

Then

$$\psi'(s) = \int_0^s \phi(v)^2\,\mathrm{d}v$$
$$\psi''(s) = \phi(s)^2.$$

Hence, denoted by

$$\|\phi\|_2 = \left(\int_0^\infty \phi(v)^2\,\mathrm{d}v\right)^{1/2}$$

we have

$$\psi(s) \leqslant s\|\phi\|_2^2$$
$$\psi'(s) \leqslant \|\phi\|_2^2.$$

Therefore

$$\begin{aligned}\frac{1}{T}\int_0^T\int_\Omega \phi(F)^2|\nabla F|^2\,\mathrm{d}V\,\mathrm{d}t &\leqslant \frac{1}{\eta T}\|\phi\|_2^2\int_\Omega F(0)\,\mathrm{d}V + \frac{1}{\eta T}M_\Phi\|\phi\|_2^2\int_0^T\int_\Omega |\boldsymbol{B}||\nabla\boldsymbol{u}|\,\mathrm{d}V\,\mathrm{d}t \\ &\leqslant \frac{1}{\eta T}\|\phi\|_2^2 M_\Phi\int_\Omega B(0)\,\mathrm{d}V + \frac{1}{\eta}\|\phi\|_2^2 M_\Phi\left(\frac{1}{T}\int_0^T\int_\Omega B^2\,\mathrm{d}V\,\mathrm{d}t\right)^{1/2} \\ &\quad\times\left(\frac{1}{T}\int_0^T\int_\Omega |\nabla\boldsymbol{u}|^2\,\mathrm{d}V\,\mathrm{d}t\right)^{1/2}.\end{aligned} \tag{45}$$

Assume now that the right hand parenthesis (whose only doubtful term is the integral of $|\nabla\boldsymbol{u}|^2$) is bounded by C. Then

$$\frac{1}{T}\int_0^T\int_\Omega \phi(F)^2|\nabla F|^2\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{C}{\eta}M_\Phi\|\phi\|_2^2 \tag{46}$$

as desired.

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